14.5

Evaluating Triple Iterated Integrals

Evaluate the integrals in Exercises 7–20.

7. $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\, dz\, dy\, dx$

8. $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dx\, dy$

9. $\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz}\, dx\, dy\, dz$

Soln:

$$\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy = \int_0^{\sqrt{2}} \int_0^{3y} z \Big|_{z=x^2+3y^2}^{z=8-x^2-y^2} dx\,dy$$

$$= \int_0^{\sqrt{2}} \int_0^{3y} 8-x^2-y^2-x^2-3y^2\, dx\,dy = \int_0^{\sqrt{2}} \int_0^{3y} 8-2x^2-4y^2\, dx\,dy$$

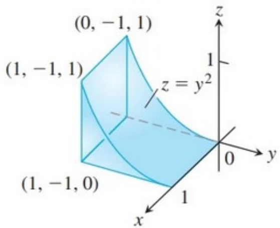
$$= \int_0^{\sqrt{2}} \left(8x - \frac{2}{3}x^3 - 4y^2x\right)\Big|_{x=0}^{x=3y} dy = \int_0^{\sqrt{2}} (24y - 18y^3 - 12y^3)\, dy$$

$$= \int_0^{\sqrt{2}} (24y - 30y^3) \Big|_{y=0}^{y=\sqrt{2}} dy = 12y^2\Big|_0^{\sqrt{2}} - \frac{30}{4}y^4\Big|_0^{\sqrt{2}} = 24 - 30 = \boxed{-6}$$

14.5

22. Here is the region of integration of the integral

$$\int_0^1 \int_{-1}^0 \int_0^{y^2} dz\, dy\, dx$$



Rewrite the integral as an equivalent iterated integral in the order

a. $dy\, dz\, dx$ **b.** $dy\, dx\, dz$

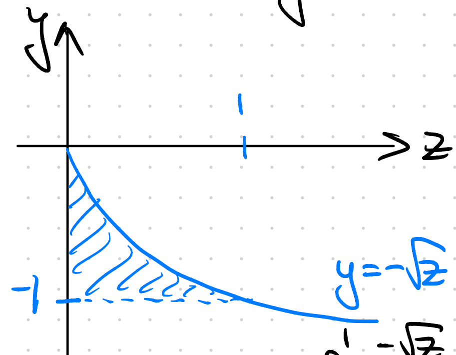
c. $dx\, dy\, dz$ **d.** $dx\, dz\, dy$

e. $dz\, dx\, dy$.

Sol'n: $0 \le x \le 1$

$-1 \le y \le 0$ $\qquad$ $-1 \le y \le -\sqrt{z}$

$0 \le z \le y^2$ $\Rightarrow$ $0 \le z \le 1$.



a) $\displaystyle\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\,dz\,dx$

b) $\displaystyle\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\,dx\,dz$

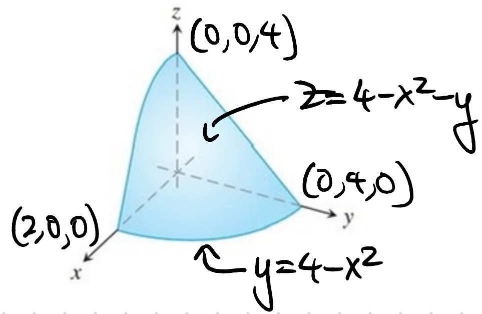
c) $\displaystyle\int_0^1 \int_{-1}^{-\sqrt{z}} \int_0^1 dx\,dy\,dz$

d) $\displaystyle\int_{-1}^0 \int_0^{y^2} \int_0^1 dx\,dz\,dy$

e) $\displaystyle\int_{-1}^0 \int_0^1 \int_0^{y^2} dz\,dx\,dy$

14.5 Find the volume of the solid

30. The region in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y$



Sol'n: First octant, so $x,y,z \geq 0$.

When $z=y=0$, $x=2$

$z=x=0$, $y=4$

$x=y=0$, $z=4$

$z=0 \Rightarrow y=4-x^2$

So
$$V=\int_0^2\int_0^{4-x^2}\int_0^{4-x^2-y} dz\,dy\,dx$$

$$=\int_0^2\int_0^{4-x^2} z\Big|_{z=0}^{z=4-x^2-y} dy\,dx$$

$$=\int_0^2\int_0^{4-x^2}(4-x^2-y)\,dy\,dx$$

$$=\int_0^2 4y-x^2y-\tfrac{1}{2}y^2\Big|_{y=0}^{y=4-x^2} dx$$

$$=\int_0^2\left(4(4-x^2)-x^2(4-x^2)-\tfrac{1}{2}(4-x^2)^2\right)dx$$

$$=\int_0^2\left(16-4x^2-\cancel{4x^2}+x^4-8+\cancel{4x^2}-\tfrac{1}{2}x^4\right)dx$$

$$=\int_0^2\left(8-4x^2+\tfrac{1}{2}x^4\right)dx=\boxed{\frac{128}{15}}$$

14.5

Average Values

In Exercises 37–40, find the average value of $F(x, y, z)$ over the given region.

37. $F(x, y, z) = x^2 + 9$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

38. $F(x, y, z) = x + y - z$ over the rectangular box in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 2$

$$\text{Volume} = \int_0^1\int_0^1\int_0^2 dz\,dy\,dx = \int_0^1\int_0^1 2\,dy\,dx = \int_0^1 2\,dx = 2$$

So, $$\text{Avg. Value} = \frac{1}{2}\int_0^1\int_0^1\int_0^2 (x+y-z)\,dz\,dy\,dx = \frac{1}{2}\int_0^1\int_0^1 \left(xz+yz-\frac{1}{2}z^2\right)\Big|_{z=0}^{z=2} dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^1 (2x+2y-2)\,dy\,dx = \frac{1}{2}\int_0^1 (2xy+y^2-2y)\Big|_{y=0}^{y=1} dx = \frac{1}{2}\int_0^1 (2x-1)\,dx$$

$$= \frac{1}{2}(x^2-x)\Big|_{x=0}^{x=1} = \boxed{0}$$

14.5

Changing the Order of Integration

Evaluate the integrals in Exercises 41–44 by changing the order of integration in an appropriate way.

41. $\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\, dy\, dz$

42. $\int_0^1 \int_0^1 \int_{x^2}^1 12xze^{zy^2}\, dy\, dx\, dz$

Sol'n: $0 \le z \le 1$

$\left.\begin{array}{l} 0 \le x \le 1 \\ x^2 \le y \le 1 \end{array}\right\} \Rightarrow \begin{array}{l} 0 \le y \le 1 \\ 0 \le x \le \sqrt{y} \end{array}$

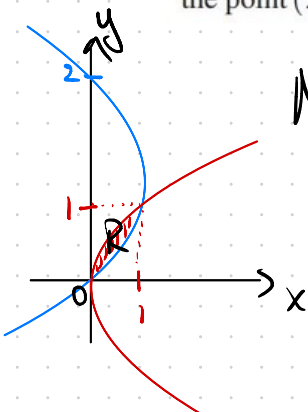
So $\int_0^1 \int_0^1 \int_{x^2}^1 12xze^{zy^2}\, dy\, dx\, dz$

$$= \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xze^{zy^2}\, dx\, dy\, dz = \int_0^1 \int_0^1 6ze^{zy^2} x^2 \Big|_{x=0}^{x=\sqrt{y}} dy\, dz = \int_0^1 \int_0^1 6zye^{zy^2}\, dy\, dz$$

$$= \int_0^1 3e^{zy^2} \Big|_{y=0}^{y=1} dz = \int_0^1 3(e^z - 1)\, dz = (3e^z - 3z)\Big|_{z=0}^{z=1} = \boxed{3e-6}$$

14.6

14. Finding a center of mass and moment of inertia Find the center of mass and moment of inertia about the x-axis of a thin plate bounded by the curves $x = y^2$ and $x = 2y - y^2$ if the density at the point (x, y) is $\delta(x, y) = y + 1$.



$$M = \iint_R \delta(x,y)\,dA = \int_0^1 \int_{y^2}^{2y-y^2} (y+1)\,dx\,dy = \int_0^1 (y+1)\left((2y-y^2) - y^2\right) dy$$

$$= \int_0^1 (-2y^3 + 2y)\,dy = \left(y^2 - \frac{1}{2}y^4\right)\Big|_{y=0}^{y=1} = \frac{1}{2}.$$

$y^2 \le x \le 2y - y^2$

$0 \le y \le 1.$

$$M_x = \iint_R y\delta(x,y)\,dA = \int_0^1 \int_{y^2}^{2y-y^2} (y^2+y)\,dx\,dy = \int_0^1 (y^2+y)(2y-2y^2)\,dy$$

$$= \int_0^1 (2y^2 - 2y^4)\,dx = \frac{4}{15}$$

$$M_y = \iint_R x\delta(x,y)\,dA = \int_0^1 \int_{y^2}^{2y-y^2} x(y+1)\,dx\,dy = \int_0^1 \frac{1}{2}(y+1)\,x^2\Big|_{x=y^2}^{x=2y-y^2} dy = \int_0^1 \frac{1}{2}(y+1)\left((2y-y^2)^2 - y^4\right) dy$$

$$= \int_0^1 (2y^2 - 2y^4)\,dy = \frac{4}{15}$$

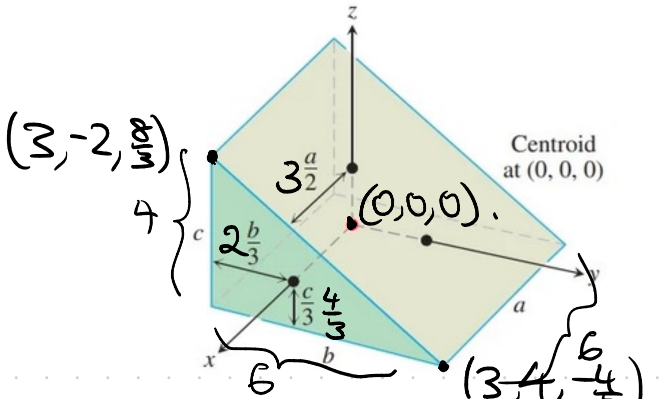
So Centre of Mass $= (\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{8}{15}, \frac{8}{15}\right)}$

Moment of Inertia about x-axis:

$$I_x = \iint_R y^2 \delta(x,y)\,dA = \int_0^1 \int_{y^2}^{2y-y^2} y^2 (y+1)\,dx\,dy = \int_0^1 (2y^3 - 2y^5)\,dy = \boxed{\frac{1}{6}}$$

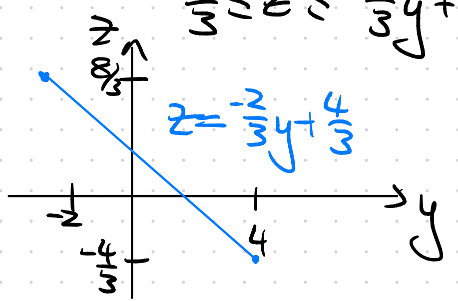
14.6

22. Moments of inertia The coordinate axes in the figure run through the centroid of a solid wedge parallel to the labeled edges. Find I_x, I_y, and I_z if $a = b = 6$, $c = 4$, and the density is $\delta(x, y, z) = 1$.



Sol'n: $-3 \le x \le 3$, $-2 \le y \le 4$

$$-\frac{4}{3} \le z \le -\frac{2}{3}y + \frac{4}{3}$$



$$I_x = \iiint_D (y^2+z^2)\,dV = \int_{-3}^{3}\int_{-2}^{4}\int_{-\frac{4}{3}}^{-\frac{2}{3}y+\frac{4}{3}} (y^2+z^2)\,dz\,dy\,dx$$

$$= \int_{-3}^{3}\int_{-2}^{4} \left(y^2 z + \frac{1}{3}z^3\right)\Big|_{z=-\frac{4}{3}}^{z=-\frac{2}{3}y+\frac{4}{3}} dy\,dx$$

$$= \int_{-3}^{3}\int_{-2}^{4} \left(y^2\left(-\frac{2}{3}y+\frac{4}{3}\right) - y^2\left(-\frac{4}{3}\right) + \frac{1}{3}\left(-\frac{2}{3}y+\frac{4}{3}\right)^3 - \frac{1}{3}\left(-\frac{4}{3}\right)^3\right) dy\,dx$$

$$= \int_{-3}^{3}\int_{-2}^{4} \left(-\frac{62}{81}y^3 + \frac{88}{27}y^2 - \frac{32}{27}y + \frac{128}{81}\right) dy\,dx$$

$$= \int_{-3}^{3} \frac{104}{3}\,dx = \boxed{208}$$

$$I_y = \int_{-3}^{3}\int_{-2}^{4}\int_{-\frac{4}{3}}^{-\frac{2}{3}y+\frac{4}{3}} (x^2+z^2)\,dz\,dy\,dx = \int_{-3}^{3}\int_{-2}^{4}\left(x^2\left(-\frac{2}{3}y+\frac{4}{3}\right) - x^2\left(-\frac{4}{3}\right) + \frac{1}{3}\left(-\frac{2}{3}y+\frac{4}{3}\right)^3 - \frac{1}{3}\left(-\frac{4}{3}\right)^3\right)dy\,dx$$

$$= \int_{-3}^{3}\int_{-2}^{4}\left(-\frac{2x^2y}{3} + \frac{8x^2}{3} - \frac{8y^3}{81} + \frac{16y^2}{27} - \frac{32}{27}y + \frac{128}{81}\right)dx = \int_{-3}^{3}\frac{4}{3}(9x^2+8)\,dx = \boxed{280}$$

$$I_z = \int_{-3}^{3}\int_{-2}^{4}\int_{-\frac{4}{3}}^{-\frac{2}{3}y+\frac{4}{3}} (x^2+y^2)\,dz\,dy\,dx = \int_{-3}^{3}\int_{-2}^{4}(x^2+y^2)\left(-\frac{2}{3}y+\frac{8}{3}\right)dy\,dx$$

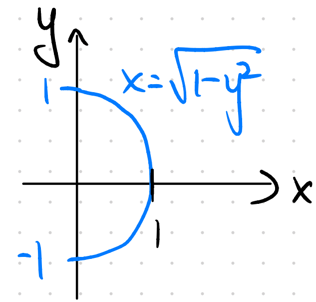
$$= \int_{-3}^{3}\int_{-2}^{4}\left(-\frac{2x^2y}{3} + \frac{8x^2}{3} - \frac{2y^3}{3} + \frac{8y^2}{3}\right)dy\,dx = \int_{-3}^{3} 12(x^2+2)\,dx = \boxed{360}$$

14.7

36. Convert the integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2+y^2)\,dz\,dx\,dy$$

to an equivalent integral in cylindrical coordinates and evaluate the result.



Sol'n: $D=\{(x,y,z)\in\mathbb{R}^3 : -1\le y\le 1,\ 0\le x\le\sqrt{1-y^2},\ 0\le z\le x\}$

In cylindrical coordinates

$D=\{(r\cos\theta,\ r\sin\theta,\ z)\in\mathbb{R}^3,\ 0\le r\le 1,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\ 0\le z\le r\cos\theta\}$ Plane $z=x$ alone.

$x^2+y^2=r^2$.

So $$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2+y^2)\,dz\,dy\,dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1}\int_{0}^{r\cos\theta} r^2\cdot r\,dz\,dr\,d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1}\int_{0}^{r\cos\theta} r^3\,dz\,dr\,d\theta$$

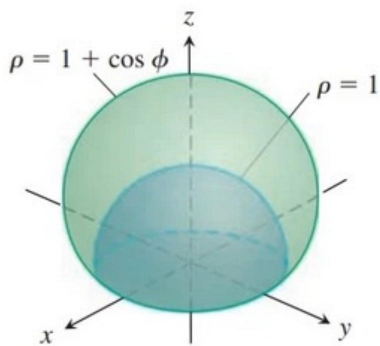
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{1} r^4 \cos\theta \, dr \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5} \cos\theta \, d\theta = \boxed{\frac{2}{5}}$$

14.7

Finding Iterated Integrals in Spherical Coordinates

In Exercises 55–60, **(a)** find the spherical coordinate limits for the integral that calculates the volume of the given solid and then **(b)** evaluate the integral.

56. The solid bounded below by the hemisphere $\rho = 1$, $z \geq 0$, and above by the surface $\rho = 1 + \cos\phi$



Sol'n: a) In spherical coordinates

$$D = \{(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi) : 1 \leq \rho \leq 1+\cos\phi,\ 0 \leq \phi \leq \tfrac{\pi}{2},\ 0 \leq \theta \leq 2\pi\}$$

$$\text{So } V = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_1^{1+\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

b) $$V=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_1^{1+\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\frac{\rho^3}{3}\sin\phi\Big|_{\rho=1}^{\rho=1+\cos\phi}d\phi\,d\theta$$

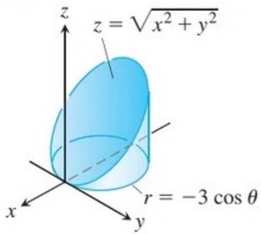
$$=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left(\frac{(1+\cos\phi)^3}{3}-\frac{1}{3}\right)\sin\phi\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\frac{1}{3}\left(3\cos\phi+3\cos^2\phi+\cos^3\phi\right)\sin\phi\,d\phi\,d\theta$$

$$=\int_0^{2\pi}-\frac{1}{3}\left(\frac{3}{2}\cos^2\phi+\frac{3}{3}\cos^3\phi+\frac{1}{4}\cos^4\phi\right)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}}d\theta=\int_0^{2\pi}\frac{11}{12}\,d\theta=\boxed{\frac{11\pi}{6}}$$

14.7 **Volumes**

Find the volumes of the solids in Exercises 65–70.

68.



Sol'n: Use cylindrical coordinates:

$$D = \left\{ (r\cos\theta, r\sin\theta, z) : 0 \le r \le -3\cos\theta,\ 0 \le z \le r,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \right\}$$

Then $V = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{-3\cos\theta} \int_0^{r} r\,dz\,dr\,d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{-3\cos\theta} r^2\,dr\,d\theta$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{3} r^3 \Big|_{r=0}^{r=-3\cos\theta} d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -9\cos^3\theta\, d\theta = -\frac{9}{4}\left(\sin\theta - \sin(3\theta)\right)\Big|_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}}$$

$$= \boxed{12}$$